ISRAEL JOURNAL OF MATHEMATICS 121 (2001), 93–111

ISOMETRIC ACTIONS OF $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ ON LORENTZ MANIFOLDS

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ABSTRACT

We prove that a locally faithful, isometric action of $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ on a connected Lorentz manifold must be a proper action. This provides an essential step toward classifying nonproper isometry groups of noncompact Lorentz manifolds.

1. Introduction

In [AS97a] and [AS97b], we gave a complete classification up to local isomorphism of the Lie groups that can appear as simply connected isometry groups of compact Lorentz manifolds. The same classification was achieved essentially simultaneously by A. Zeghib ([Zeghib95b] and [Zeghib95a]). In this paper, we take up the investigation of isometric group actions on *noncompact* Lorentz manifolds. We consider actions that have nontrivial dynamics in the sense that the

* The first author was supported in part by NSF grant DMS-9703480.
Received March 7, 1999

action is *nonproper*. In [Kowalsky96], N. Kowalsky investigated nonproper isometric actions of *simple* Lie groups on Lorentz manifolds, and made several fundamental discoveries. As a first step toward studying isometric actions of general nonsimple groups, we consider the special case $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$, $n \geq 3$. In other papers ([Adams98a], [Adams99a], [Adams99b], [Adams99d]), we use the techniques developed here to study more general situations. One may hope, using these techniques, to give a list of the connected Lie groups that admit a nonproper, faithful isometric action on a connected Lorentz manifold. (See Theorem 1.1 and Theorem 1.2 of [Adams99d].)

Here, we prove (Theorem 10.1) that a locally faithful action of $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$, $n \geq 3$, by isometric transformations of a connected Lorentz manifold M must be a proper action.

Note that if A is a Lie group and if there is an isometric action of $\mathrm{Aut}(A)^0 \ltimes A$ on a Lorentz manifold such that the restriction to A is nonproper, then *any* connected Lie group with a normal subgroup A' isomorphic to A admits an isometric action such that A' is nonproper. This fact is a consequence of Corollary 4.4 in [Adams99c]. Since $\mathrm{Aut}(\mathbb{R}^n)$ is $\mathrm{GL}(n, \mathbb{R})$, it is reasonable, for any positive integer n, to ask whether $\mathrm{GL}(n, \mathbb{R})^0 \ltimes \mathbb{R}^n$ has an action such that $\mathbb{R}^n$ is nonproper. If there were such an action, then the restriction to $\mathrm{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ would still be nonproper, and this paper shows that that is impossible, for $n \geq 3$. In [AS99], we refine our methods to extend this result to *conformal* actions. By Theorem 1.1 of [Adams99d], it follows that any locally faithful isometric action of $\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ on a connected Lorentz manifold is proper. The methods used in [Adams99d] are far more delicate than the ones used here.

We proceed to an outline of the proof that a locally faithful action of $G = \mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$, $n \geq 4$, by isometries of a connected Lorentz manifold M is proper: We suppose that the action of G on M is nonproper, and argue to obtain a contradiction.

Let F be the natural equivariant map from M to the vector space of symmetric bilinear forms on $\mathfrak{g}$. Following ideas of [Kowalsky96], we use F to establish that $\mathbb{R}^n$ is lightlike at some point $m_0 \in M$, so that some codimension one subgroup S in $\mathbb{R}^n$ must stabilize m_0.

On the other hand, the Adjoint action of $\mathbb{R}^n$ on $\mathfrak{g}/\mathbb{R}^n$ is trivial, and so preserves all symmetric bilinear forms. As a result, there are many $\mathbb{R}^n$-invariant symmetric bilinear forms on $\mathfrak{g}$ whose kernel contains $\mathbb{R}^n$. Any of these forms will give rise to a G-invariant form on $\mathfrak{g}/\mathbb{R}^n$. Consequently, the study of F cannot be used to prove that any element of $\mathfrak{sl}_n(\mathbb{R})$ is lightlike at any point of M. In particular, it cannot establish that any element of $\mathrm{SL}_n(\mathbb{R})$ stabilizes a point.

To proceed, we use a new idea: Let $d := \dim M$ and $n_0 := \dim S$. The action of G on M gives, for every element of $\mathfrak{g}$, an isometric Killing vector field on M. We use 1-jets of these vector fields (viewed in exponential normal coordinates) to construct a Lie subalgebra of $\mathfrak{so}(1, d-1)$ that admits a Lie algebra surjection onto $\mathfrak{sl}_{n_0}(\mathbb{R})$. As $n \geq 4$, we have $n_0 = n - 1 \geq 3$, so the split rank of $\mathfrak{sl}_{n_0}(\mathbb{R})$ is greater than the split rank of $\mathfrak{so}(1, d-1)$, giving a contradiction. A slightly improved version of this argument which works for $n \geq 3$.

ACKNOWLEDGEMENT: We appreciate the careful reading of the referee. The first author would like to thank V. Reiner and J. Roberts for their help (see §7) in analyzing some of the representation theoretic questions in this paper.

An early version of this paper was prepared while the authors were participants of the Research-in-Pairs Program at Oberwolfach, sponsored by Volkswagon-Stiftung. The working environment was ideal, and we appreciated very much the hospitality of our hosts.

2. Global definitions

Throughout this paper, "vector space" will always mean "real vector space", "manifold" will always mean "real manifold", and "Lie group" will always mean "real Lie group". All tensors will be assumed to be smooth (C^∞).

Let V be a vector space. If $v \in V$ and if v_i is a sequence in V, then v_i **converges in direction to** v if $\mathbb{R}v_i \to \mathbb{R}v$ in the topological space of linear subspaces of V. We denote by $\mathrm{SBF}(V)$ the collection of symmetric bilinear forms on V.

Let S and T be tensors on a manifold M defined near a point $m \in M$. Let k be a positive integer. We say that S **vanishes to order** k **at** m if S vanishes at m and if, for all $l \in \{1, \ldots, k\}$, for all vector fields $X_1, \ldots, X_l$ on M, we have that $(L_{X_1} L_{X_2} \cdots L_{X_l})(S)$ vanishes at m. We say that S **and** T **agree to order** k **at** m if $S - T$ vanishes to order k at m.

We will say that a vector field X on $\mathbb{R}^n$ is **homogeneous of degree** k if there are homogeneous polynomials $p_1, \ldots, p_n$ of degree k such that $X = \sum_{i=1}^n p_i(\partial/\partial x_i)$. We say that X is **constant** if it is homogeneous of degree 0 (i.e., a constant linear combination of the coordinate vector fields $\partial/\partial x_i$); **linear** if it is homogeneous of degree 1; and **quadratic** if it is homogeneous of degree 2.

A vector field X on a vector space V defined near zero is **constant** (resp. **linear**, **quadratic**) if there is an isomorphism between V and $\mathbb{R}^{\dim V}$ under which X corresponds to a constant (resp. linear, quadratic) vector field near zero.

A **quadratic differential** on a manifold is a smoothly varying system of quadratic forms, one on each tangent space of the manifold. A quadratic differential on a vector space V is **constant** if, for all $v \in V$, it is invariant under $w \mapsto v + w : V \to V$.

Let V be a vector space with quadratic form Q. Then $\mathrm{SO}(Q)$ denotes the Lie group of orientation-preserving linear transformations of V preserving Q, and $\mathfrak{so}(Q)$ denotes the Lie algebra of $\mathrm{SO}(Q)$. The Lie algebra $\mathfrak{so}(Q)$ can be identified with the Lie algebra of linear vector fields X on V such that the flow of X preserves Q.

If X is a locally compact first-countable topological space and if x_i is a sequence in X, then x_i **goes to infinity** if, for every compact set $K \subseteq X$, for all but a finite number of i, we have $x_i \notin K$. We write $x_i \to \infty$ to indicate that x_i goes to infinity.

A continuous action of a locally compact first-countable group G on a locally compact first-countable topological space X is **proper** if, for every compact $K \subseteq X$, the set $\{g \in G \,|\, gK \cap K \neq \emptyset\}$ is compact. A sequence g_i in G is a **nonproper sequence** if both

1. $g_i \to \infty$ in G; and
2. there exists a sequence $\{x_i\}$ in X such that $\{x_i\}$ and $\{g_i x_i\}$ are both convergent sequences in X.

Note that the G-action is nonproper if and only if there is a nonproper sequence in G. If $\{g_i\}$ is a nonproper sequence, then so is $\{g_i^{-1}\}$.

If g_i and h_i are sequences in a locally compact first-countable group G, then h_i **is a bounded perturbation of** g_i if there exist two convergent sequences $\{k_i\}$ and $\{l_i\}$ in G such that $h_i = k_i g_i l_i$ for all i.

Let G be a connected semisimple Lie group with finite center and let $\mathfrak{a}$ be a maximal split torus in $\mathfrak{g}$. Then $\Gamma(\mathfrak{g}, \mathfrak{a})$ will denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. For each $\alpha \in \Gamma$, let $\mathfrak{g}_\alpha$ be the root space of α. For $A \in \mathfrak{a}$, let $\Gamma_A^+ = \Gamma_A^+(\mathfrak{g}, \mathfrak{a}) := \{\alpha \in \Gamma \,|\, \alpha(A) > 0\}$ and $\mathfrak{n}_A^+(\mathfrak{g}, \mathfrak{a}) := \bigoplus_{\alpha \in \Gamma_A} \mathfrak{g}_\alpha$.

For the remainder of this paper, fix a Lie group G and let G act by isometries of a connected pseudo-Riemannian manifold (M, γ). For $m \in M$, let $B_m \in \mathrm{SBF}(\mathfrak{g})$ be the pullback of γ_m by the differential $\mathfrak{g} \to T_m M$ at e of the orbit map $g \mapsto gm$: $G \to M$. If $V \subseteq \mathfrak{g}$ is a subspace and if $m \in M$, then V is **isotropic at** m if V is B_m-isotropic, i.e., $B_m|V$ is zero. If $V \subseteq \mathfrak{g}$ is a subspace, then V is **somewhere isotropic** if there exists $m \in M$ such that V isotropic at m. For

$m \in M$, let $G_m := \mathrm{Stab}_G(m)$ and let $\mathfrak{g}_m$ denote the Lie algebra of G_m.

Define $E_m: G \to M$ by $E_m(g) = gm$ and let $e_m: \mathfrak{g} \to T_m M$ be the differential at 1_G of E_m. Let $B_m := e_m^*(\gamma_m)$, so B_m is a symmetric bilinear form on $\mathfrak{g}$.

For $X \in \mathfrak{g}$, let X_M be the vector field on M corresponding to X. For $X \in \mathfrak{g}$ and $m \in M$, let $X_m := (X_M)_m \in T_m M$ denote the value of X_M at the point m.

3. Nonproper isometric actions

The results in this section are essentially due to Kowalsky.

LEMMA 3.1: *Let W_i and X_i be convergent sequences in $\mathfrak{g}$ and let $Y, Z \in \mathfrak{g}$. Let g_i be a sequence in G, let m_i be a convergent sequence in M and let $m \in M$. Assume that $g_i m_i \to m$ in M. Assume that $\{(\mathrm{Ad}\, g_i)W_i\}_i$ goes to infinity in $\mathfrak{g}$, but converges in direction to Y. Assume that $(\mathrm{Ad}\, g_i)X_i$ does not converge to zero in $\mathfrak{g}$, and converges in direction to Z as $i \to \infty$. Then $B_m(Y, Z) = 0$.*

Proof: For all i, let $Y_i := (\mathrm{Ad}\, g_i)W_i$ and let $Z_i := (\mathrm{Ad}\, g_i)X_i$. Choose a sequence t_i in the interval $(0, \infty)$ such that $Y_i/t_i \to Y$. Choose a sequence u_i in $(0, \infty)$ such that $u_i Z_i \to Z$.

Since $Y_i \to \infty$, it follows that $t_i \to \infty$ in the interval $(0, \infty)$. Since Z_i does not converge to zero, u_i does not approach ∞ in the interval $(0, \infty)$. Passing to a subsequence, we may assume that u_i is bounded above. Choose $K \in (0, \infty)$ such that, for all i, $u_i \leq K$.

Now $B_{g_i m_i}(Y_i/t_i, u_i Z_i) \to B_m(Y, Z)$, so it suffices to show that

$$(u_i/t_i)[B_{g_i m_i}(Y_i, Z_i)] \to 0.$$

Since $t_i \to \infty$ and since, for all i, we have $u_i \leq K$, we conclude that $u_i/t_i \to 0$. It therefore suffices to show that $B_{g_i m_i}(Y_i, Z_i)$ is bounded.

For all i, we have

$$\begin{aligned} B_{g_i m_i}(Y_i, Z_i) &= B_{m_i}\big((\mathrm{Ad}\, g_i)^{-1} Y_i, (\mathrm{Ad}\, g_i)^{-1} Z_i \big) \\ &= B_{m_i}(W_i, X_i). \end{aligned}$$

Since m_i, W_i and X_i are all convergent, it follows that $B_{g_i m_i}(Y_i, Z_i)$ is bounded, as desired. ∎

COROLLARY 3.2: *Let $\{X_i^1\}_i, \ldots, \{X_i^k\}_i$ be k convergent sequences in $\mathfrak{g}$ and let $Y^1, \ldots, Y^k \in \mathfrak{g}$. Let g_i be a nonproper sequence in G. Assume, for all $j \in$*

$\{1,\ldots,k\}$, that $\{(\mathrm{Ad}\, g_i)X_i^j\}_i$ is divergent in $\mathfrak{g}$, but converges in direction to Y^j. Then the span of $Y^1,\ldots,Y^k$ is somewhere isotropic.

Proof: Choose a convergent sequence m_i in M and $m \in M$ such that $g_i m_i \to m$. By Lemma 3.1, for all $j, j' \in \{1,\ldots,k\}$, we have

$$B_m(Y^j, Y^{j'}) = 0.$$

Thus the span of $Y^1,\ldots,Y^k$ is B_m-isotropic. ∎

COROLLARY 3.3: *Let $S \subseteq \mathfrak{g}$ be a subset. Let g_i be a nonproper sequence in G. Assume, for all $X \in S$, that $(\mathrm{Ad}\, g_i)X \to \infty$ in $\mathfrak{g}$. Assume, for all $X \in S$, that $\{(\mathrm{Ad}\, g_i)X\}_i$ converges in direction to a vector $Y_X \in \mathfrak{g}$ as $i \to \infty$. Then the span of $\{Y_X \mid X \in S\}$ is somewhere isotropic.*

Proof: Choose a positive integer k and $X^1,\ldots,X^k \in S$ such that the span of $Y_{X^1},\ldots,Y_{X^k}$ is the same as the span of $\{Y_X \mid X \in S\}$. We wish to show that the span of $Y_{X^1},\ldots,Y_{X^k}$ is somewhere isotropic.

For all i, for all $j \in \{1,\ldots,k\}$, set $X_i^j := X^j$ and $Y^j := Y_{X^j}$. The result now follows from Corollary 3.2. ∎

COROLLARY 3.4: *Let $S \subseteq \mathfrak{g}$ be a subset. Let g_i be a nonproper sequence in G. Assume, for all $X \in S$, that $(\mathrm{Ad}\, g_i)X \to 0$ in $\mathfrak{g}$. Then the span of S is somewhere isotropic.*

Proof: Choose a positive integer k and $Y^1,\ldots,Y^k \in S$ such that the span of $Y^1,\ldots,Y^k$ is the same as the span of S.

For all $j \in \{1,\ldots,k\}$, we have $(\mathrm{Ad}\, g_i)Y^j \to 0$ as $i \to \infty$. By passing to a subsequence, we may assume, for all $j \in \{1,\ldots,k\}$, that $\{(\mathrm{Ad}\, g_i)Y^j\}_i$ converges in direction.

For all $j \in \{1,\ldots,k\}$, choose a sequence $\{t_i^j\}_i$ in the interval $(0,\infty)$ such that the sequence $\{t_i^j(\mathrm{Ad}\, g_i)Y^j\}_i$ converges to a nonzero vector X^j in $\mathfrak{g}$; then $t_i^j \to \infty$ in the interval $(0,\infty)$.

For all $j \in \{1,\ldots,k\}$, for all i, let $X_i^j := t_i^j(\mathrm{Ad}\, g_i)Y^j$. For all $j \in \{1,\ldots,k\}$, we know that $X_i^j \to X^j \neq 0$ as $i \to \infty$.

For all $j \in \{1,\ldots,k\}$, for all i, we have $(\mathrm{Ad}\, g_i^{-1})(X_i^j) = t_i^j Y^j$. So, for $j \in \{1,\ldots,k\}$, the sequence $\{(\mathrm{Ad}\, g_i^{-1})(X_i^j)\}_i$ goes to infinity, but converges in direction to Y^j. The result follows from Corollary 3.2. ∎

Recall that $\mathfrak{n}_A^+(\mathfrak{g},\mathfrak{a})$ is defined in §2.

LEMMA 3.5: *Assume that G is a connected semisimple Lie group with finite center. Let $\mathfrak{a}$ be a maximal split torus in $\mathfrak{g}$. If the action of G on M is nonproper, then there exists $A_0 \in \mathfrak{a}\backslash\{0\}$ such that $\mathfrak{n}^+_{A_0}(\mathfrak{g}, \mathfrak{a})$ is somewhere isotropic.*

Proof: Let A the the connected subgroup of G corresponding to $\mathfrak{a}$, and let K be a maximal compact subgroup of G. Using the Cartan decomposition $G = KAK$, we see that any nonproper sequence in G has a bounded perturbation which is contained in A. A bounded perturbation of a nonproper sequence has a nonproper subsequence, so there is a nonproper sequence $\{a_i\}_i$ in A.

For each i, choose $A_i \in \mathfrak{a}$ such that $\exp(A_i) = a_i$. Passing to a subsequence, we assume that A_i converges in direction to $A_0 \in \mathfrak{a}\backslash\{0\}$.

The result now follows from Corollary 3.3, with $S := \bigcup_{\alpha\in\Gamma^+_{A_0}} \mathfrak{g}_\alpha$. ∎

4. Isotropic subspaces of $\mathfrak{g}$

Fix $m_0 \in M$. Let $E_0 := E_{m_0}$ and let $e_0 := e_{m_0}$.

LEMMA 4.1: *If $V \subseteq \mathfrak{g}$ is a subspace isotropic at $m_0 \in M$, then $\mathfrak{g}_{m_0}$ contains a codimension one subspace of V.*

Proof: Since V is B_{m_0}-isotropic, it follows that $e_0(V)$ is isotropic in $T_{m_0}M$, and therefore has dimension at most one. Thus the kernel of $e_0|V$ has codimension at most one in V, as desired. ∎

LEMMA 4.2: *If $X \in \mathfrak{g}_{m_0}$, then*

$$[(\mathrm{ad}_\mathfrak{g} X)(\mathfrak{g})] \cap [(\mathrm{ad}_\mathfrak{g} X)^{-1}(\mathfrak{g}_{m_0})]$$

is isotropic at m_0.

Proof: Fix $Y \in [(\mathrm{ad}_\mathfrak{g} X)(\mathfrak{g})] \cap [(\mathrm{ad}_\mathfrak{g} X)^{-1}(\mathfrak{g}_{m_0})]$. We wish to show that $B_{m_0}(Y,Y) = 0$.

As $Y \in (\mathrm{ad}_\mathfrak{g} X)(\mathfrak{g})$, fix $W \in \mathfrak{g}$ such that $Y = (\mathrm{ad}_\mathfrak{g} X)W = [X, W]$. Since $X \in \mathfrak{g}_{m_0}$, we have $B_{m_0}([X,W],Y) + B_{m_0}(W,[X,Y]) = 0$. Since $Y \in (\mathrm{ad}_\mathfrak{g} X)^{-1}(\mathfrak{g}_{m_0})$, we get $[X,Y] = (\mathrm{ad}_\mathfrak{g} X)Y \in \mathfrak{g}_{m_0} \subseteq \ker(B_{m_0})$. Then

$$B_{m_0}(Y,Y) = B_{m_0}([X,W],Y) = -B_{m_0}(W,[X,Y]) = 0. \qquad ∎$$

5. Actions of $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$

Fix an integer $n \geq 3$. Assume that $G = \mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$. Assume that (M, γ) is Lorentz. Assume that G acts nonproperly on M. (We will eventually show that this is impossible.)

Let $K := \mathrm{SO}(n)$. Let A be the maximal split torus in $\mathrm{SL}_n(\mathbb{R})$ consisting of $n \times n$ diagonal matrices with positive diagonal entries and determinant one.

LEMMA 5.1: *The action of $A \ltimes \mathbb{R}^n$ on M is nonproper.*

Proof: As $\mathrm{SL}_n(\mathbb{R}) = KAK$, we get $G = K(A \ltimes \mathbb{R}^n)K$. So any sequence in G has a bounded perturbation in $A \ltimes \mathbb{R}^n$. A bounded perturbation of a nonproper sequence has a nonproper subsequence. ∎

LEMMA 5.2: *Let $X \in \mathfrak{sl}_n(\mathbb{R})\backslash\{0\}$ and let $v \in \mathbb{R}^n$. Let $m_0 \in M$. Assume that every row of X vanishes except the first. Assume that every entry of v vanishes except possibly the first. Then $X + v \notin \mathfrak{g}_{m_0}$.*

Proof: Assume, for a contradiction, that $X + v \in \mathfrak{g}_{m_0}$.

Let $\tilde{X} : \mathbb{R}^n \to \mathbb{R}^n$ denote the endomorphism corresponding to X. Because every entry of v vanishes except the first, because every row of X vanishes except the first and because $X \neq 0$, choose $w \in \mathbb{R}^n$ such that $\tilde{X}w = v$. Then

$$\begin{aligned}(\mathrm{Ad}\, w)(X + v) &= ((\mathrm{Ad}\, w)X) + v = X + [w, X] + v \\ &= X - [X, w] + v = X - \tilde{X}w + v \\ &= X - v + v = X.\end{aligned}$$

Replacing m_0 by wm_0 and replacing $X + v$ by $(\mathrm{Ad}\, w)(X + v)$, we may assume that $v = 0$.

Then $H := \{\exp(tX) \,|\, t \in \mathbb{R}\} \subseteq G_{m_0}$. As H is a noncompact subgroup of $\mathrm{SL}_n(\mathbb{R})$, we conclude that the $\mathrm{SL}_n(\mathbb{R})$ action on M is nonproper. As $n \geq 3$, this contradicts [Kowalsky96]. ∎

LEMMA 5.3: *The action of $\mathbb{R}^n$ on M is nonproper.*

Proof: By Lemma 5.1, there exists a nonproper sequence g_i in $A \ltimes \mathbb{R}^n$.

For all i, choose a_i in A and v_i in $\mathbb{R}^n$ such that $g_i = a_i v_i$.

If $\{a_i\}$ has a convergent subsequence, then after passing to this subsequence and making a bounded perturbation we conclude that v_i is a nonproper sequence; this would imply that $\mathbb{R}^n$ is nonproper on M, and we would be done. We therefore assume that $a_i \to \infty$ in A, and aim for a contradiction.

For each i, for each $j \in \{1, \ldots, n\}$, let a_i^j be the (j,j) entry of a_i and let v_i^j be the jth entry of v_i; by definition of A, we have $a_i^j > 0$.

For all $j, k \in \{1, \ldots, n\}$, let E_{jk} denote the matrix with a one in the (j,k) entry and zeroes everywhere else and let u_j denote the vector in $\mathbb{R}^n$ with a one in the jth entry and zeroes everywhere else.

Reordering coordinates, we may assume that $a_i^1 \to \infty$ and $a_i^2 \to 0$.

If $a_i^3 \to \infty$, then both $(\mathrm{Ad}\, g_i)u_1 = a_i^1 u_1$ and $(\mathrm{Ad}\, g_i)u_3 = a_i^3 u_3$ go to infinity and converge in direction to u_1 and u_3, respectively. By Corollary 3.3, we conclude that the span $\mathbb{R}u_1 + \mathbb{R}u_3$ is somewhere isotropic. By Lemma 4.1, a codimension one subspace in $\mathbb{R}u_1 + \mathbb{R}u_3$ is contained in the stabilizer of some point, so $\mathbb{R}^n$ acts nonproperly and we are done.

We assume then that a_i^3 does not go to infinity. Passing to a subsequence, we may assume that a_i^3 is bounded.

For all i, we have $(\mathrm{Ad}\, g_i)E_{12} = (a_i^1/a_i^2)E_{12} - a_i^1 v_i^2 u_1$, which, after passing to a subsequence, converges in direction. Choose $X \in \mathbb{R}E_{12}$ and $u \in \mathbb{R}u_1$ such that $(\mathrm{Ad}\, g_i)E_{12}$ converges in direction to $X + u$.

Similarly, for all i, we have $(\mathrm{Ad}\, g_i)E_{13} = (a_i^1/a_i^3)E_{13} - a_i^1 v_i^3 u_1$, which goes to infinity, and, after passing to a subsequence, converges in direction. Choose $Y \in \mathbb{R}E_{13}$ and $v \in \mathbb{R}u_1$ such that $(\mathrm{Ad}\, g_i)E_{13}$ converges in direction to $Y + v$.

We consider first the case where $X \neq 0$. Because E_{12} and E_{13} are linearly independent, $X + u$ and $Y + v$ are linearly independent in $\mathfrak{sl}_n(\mathbb{R}) \ltimes \mathbb{R}^n$. It follows from Corollary 3.3 and Lemma 4.1 that we can choose $m_0 \in M$ and $s, t \in \mathbb{R}$ such that $s(X + u) + t(Y + v) \in \mathfrak{g}_{m_0}\backslash\{0\}$. Let $Z := sX + tY$ and $w := su + tv$.

Then $Z + w \in \mathfrak{g}_{m_0}\backslash\{0\}$ and $Z \in \mathbb{R}E_{12} + \mathbb{R}E_{13}$ and $w \in \mathbb{R}u_1$. So all but the first row of Z vanishes and all but possibly the first entry of w vanishes. In this case, by Lemma 5.2, we must have $Z = 0$, and obtain a nontrivial (hence noncompact) stabilizer for the $\mathbb{R}^n$-action. Thus $\mathbb{R}^n$ acts nonproperly, as desired, provided that $X \neq 0$.

A similar argument will work in the case where $Y \neq 0$. We may therefore assume that $X = Y = 0$.

Since $(a_i^1/a_i^3)E_{13} - a_i^1 v_i^3 u_1$ converges in direction to $Y + v = v \in \mathbb{R}u_1$, it follows that $(a_i^1/a_i^3)/(a_i^1 v_i^3) \to 0$.

As $X = 0$, it follows that $X + u = u \in \mathbb{R}u_1$. So $(\mathrm{Ad}\, g_i)E_{12}$ converges in direction to u_1.

Now, for all i, we have $(\mathrm{Ad}\, g_i)E_{23} = (a_i^2/a_i^3)E_{23} - a_i^2 v_i^3 u_2$, and, since

$$(a_i^2/a_i^3)/(a_i^2 v_i^3) = (a_i^1/a_i^3)/(a_i^1 v_i^3) \to 0,$$

we see that $(\mathrm{Ad}\, g_i)(E_{23})$ converges in direction to u_2.

We therefore conclude, from Corollary 3.3 (with $S := \{E_{12}, E_{23}\}$), that the span of u_1 and u_2 is somewhere isotropic.

By Lemma 4.1, this implies that a noncompact subgroup of $\mathbb{R}^n$ stabilizes some point of M. Thus $\mathbb{R}^n$ acts nonproperly on M. ∎

6. Codimension one stabilizers in $\mathbb{R}^n$

Fix an integer $n \geq 2$. Assume that $G = \mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$. Assume that (M, γ) is Lorentz. Assume that $\mathbb{R}^n$ acts nonproperly on M. (We will eventually show that this is impossible.)

LEMMA 6.1: *The subalgebra $\mathbb{R}^n$ of $\mathfrak{g}$ is somewhere isotropic.*

Proof: By assumption, there is a nonproper sequence v_i in $\mathbb{R}^n$. Fix a norm on $\mathbb{R}^n$. Choose sequences $\{w_i\}$ in the unit sphere of $\mathbb{R}^n$ and $\{t_i\}$ in $(0, \infty)$ such that $t_i \to \infty$ and such that $v_i = t_i w_i$. Passing to a subsequence, we may assume that there exists w_∞ on the unit sphere of $\mathbb{R}^n$ such that $w_i \to w_\infty$.

For all $X \in \mathfrak{sl}_n(\mathbb{R})$, let $\tilde{X} : \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding endomorphism; we then have

$$(\mathrm{Ad}\, v_i)X = X + [v_i, X] = X - \tilde{X} v_i = X - t_i(\tilde{X} w_i).$$

For all $X \in P := \{X \in \mathfrak{sl}_n(\mathbb{R}) |\ \tilde{X} w_\infty \neq 0\}$, we conclude that the sequence $\{(\mathrm{Ad}\, v_i)X\}_i$ goes to infinity, but converges in direction to $\tilde{X} w_\infty$.

Note that $\mathbb{R}^n$ is the span of $\{\tilde{X} w_\infty \,|\, X \in \mathfrak{sl}_n(\mathbb{R})\}$, and therefore is also the span of $S := \{\tilde{X} w_\infty \,|\, X \in P\}$. Thus, by Corollary 3.3, we conclude that $\mathbb{R}^n$ is somewhere isotropic, as desired. ∎

7. Some representation-theoretic results

Roughly speaking, the object of this section is to show that the Lie algebra $\mathfrak{so}(1, d-1) \ltimes \mathbb{R}^d$ does not contain $\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ as a subalgebra. More precisely, the following lemma, together with Lemma 7.2, shows that there is no nonzero linear map from $\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ to $\mathfrak{so}(1, d-1) \ltimes \mathbb{R}^d$ that preserves the Lie bracket of elements of $\mathfrak{sl}_2(\mathbb{R})$ with elements of $\mathbb{R}^2$.

LEMMA 7.1: *Let $\mathfrak{b}$ and $\mathfrak{h}$ be Lie algebras and let $W \subseteq \mathfrak{b}$ be a subspace that is not contained in any proper Lie subalgebra of $\mathfrak{b}$. Let ϕ: $W \to \mathfrak{h}$ be a linear map. Let $\mathfrak{h}_0$ denote the smallest Lie subalgebra of $\mathfrak{h}$ that contains $\phi(W)$. Let V be a vector space, and let ρ: $\mathfrak{b} \to \mathfrak{gl}(V)$ be a representation. For all $X \in \mathfrak{b}$,* *let* $\tilde{X} := \rho(X)\colon V \to V$ *be the endomorphism corresponding to* X. *Define* $\tilde{\mathfrak{b}} := \rho(\mathfrak{b}) \subseteq \mathfrak{gl}(V)$. *Let* $\psi\colon V \to \mathfrak{h}$ *be a linear map. Assume, for all* $X \in W$, $U \in V$, *that*

$$[\phi(X), \psi(U)] = \psi(\tilde{X}U). \tag{7.1}$$

Then

1. *if* $\rho\colon \mathfrak{b} \to \mathfrak{gl}(V)$ *is irreducible and if* $\psi \neq 0$, *then* $\psi\colon V \to \mathfrak{h}$ *is injective;*
2. *if* ψ *is injective, then* $\tilde{\mathfrak{b}}$ *is a Lie quotient of* $\mathfrak{h}_0$, *i.e., there exists a surjective Lie algebra homomorphism* $\sigma\colon \mathfrak{h}_0 \to \tilde{\mathfrak{b}}$; *and*
3. *if* ψ *is injective and if* $\tilde{\mathfrak{b}}$ *is semisimple, then there exists a Lie algebra homomorphism* $\phi'\colon \mathfrak{b} \to \mathfrak{h}_0$ *such that, for all* $X \in \mathfrak{b}$, *for all* $U \in V$, *we have*

$$[\phi'(X), \psi(U)] = \psi(\tilde{X}U). \tag{7.2}$$

Proof of Lemma 7.1: *Proof of 1:* Assume that ρ is irreducible and $\psi \neq 0$. Suppose $U_0 \in V$ and $\psi(U_0) = 0$. By (7.1), we conclude, for all $X \in \mathfrak{b}$, that $(\rho(X))U_0 = \tilde{X}U_0 \in \ker(\psi)$. Since $\rho\colon \mathfrak{b} \to \mathfrak{gl}(V)$ is irreducible, the linear span of $\{(\rho(X))U_0 \,|\, X \in \mathfrak{b}\}$ is either 0 or V. Since $\psi = 0$, we conclude that $U_0 = 0$.

Proof of 2: We now assume that ψ is injective.

Let $\mathcal{W} := \phi(W)$. Then no proper Lie subalgebra of $\mathfrak{h}_0$ contains $\mathcal{W}$. Let $\mathcal{V} := \psi(V)$. Let $\tilde{W} := \rho(W)$.

The map $\psi\colon V \to \mathcal{V}$ is an isomorphism of vector spaces and therefore induces an isomorphism $\Psi\colon \mathfrak{gl}(V) \to \mathfrak{gl}(\mathcal{V})$ of Lie algebras. Then, for all $\mathcal{X} \in \mathfrak{gl}(V)$, for all $U \in V$, we have

$$[\Psi(\mathcal{X})][\psi(U)] = \psi(\mathcal{X}U). \tag{7.3}$$

By (7.1), we know, for all $X \in W$, that $\mathrm{ad}_{\mathfrak{h}}(\phi(X))\colon \mathfrak{h} \to \mathfrak{h}$ preserves $\mathcal{V}$. From (7.1) and (7.3), we get

$$\mathrm{ad}_{\mathfrak{h}}(\phi(X))|\mathcal{V} = \Psi(\tilde{X}). \tag{7.4}$$

Let $\mathfrak{n}_{\mathfrak{h}}(\mathcal{V})$ denote the normalizer in $\mathfrak{h}$ of $\mathcal{V}$. Then $\mathfrak{n}_{\mathfrak{h}}(\mathcal{V})$ is a Lie subalgebra of $\mathfrak{h}$ and, from (7.1), it follows that $\mathcal{W} \subseteq \mathfrak{n}_{\mathfrak{h}}(\mathcal{V})$. Since no proper Lie subalgebra of $\mathfrak{h}_0$ contains $\mathcal{W}$, it follows that $\mathfrak{h}_0 \subseteq \mathfrak{n}_{\mathfrak{h}}(\mathcal{V})$.

Define $\tau\colon \mathfrak{n}_{\mathfrak{h}}(\mathcal{V}) \to \mathfrak{gl}(\mathcal{V})$ by $\tau(Y) = (\mathrm{ad}_{\mathfrak{h}} Y)|\mathcal{V}$. Then τ is a Lie algebra homomorphism.

By (7.4), we conclude, for all $X \in W$, that $\tau(\phi(X)) = \Psi(\tilde{X})$. Thus $\tau(\mathcal{W}) = \Psi(\tilde{W})$, which implies that $\mathcal{W} \subseteq \tau^{-1}(\Psi(\tilde{\mathfrak{b}}))$. Since no proper Lie subalgebra of $\mathfrak{h}_0$ contains $\mathcal{W}$, it follows that $\mathfrak{h}_0 \subseteq \tau^{-1}(\Psi(\tilde{\mathfrak{b}}))$, which implies that $\tau(\mathfrak{h}_0) \subseteq \Psi(\tilde{\mathfrak{b}})$.

Since $\Psi(\rho(W)) = \Psi(\tilde{W}) = \tau(W) \subseteq \tau(\mathfrak{h}_0)$, we conclude that

$$W \subseteq \rho^{-1}(\Psi^{-1}(\tau(\mathfrak{h}_0))).$$

By assumption, no proper Lie subalgebra of $\mathfrak{b}$ contains W, so

$$\mathfrak{b} \subseteq \rho^{-1}(\Psi^{-1}(\tau(\mathfrak{h}_0))).$$

Then $\Psi(\tilde{\mathfrak{b}}) = \Psi(\rho(\mathfrak{b})) \subseteq \tau(\mathfrak{h}_0)$. We have already established that $\tau(\mathfrak{h}_0) \subseteq \Psi(\tilde{\mathfrak{b}})$, so $\tau(\mathfrak{h}_0) = \Psi(\tilde{\mathfrak{b}})$. It follows that $(\Psi^{-1} \circ \tau)(\mathfrak{h}_0) = \tilde{\mathfrak{b}}$, so $\sigma := (\Psi^{-1} \circ \tau)|\mathfrak{h}_0 : \mathfrak{h}_0 \to \mathfrak{gl}(V)$ is the desired map.

Proof of 3: We now assume that ψ is injective and $\tilde{\mathfrak{b}}$ is semisimple.

Let σ: $\mathfrak{h}_0 \to \tilde{\mathfrak{b}}$ be the surjective homomorphism constructed above. As $\tilde{\mathfrak{b}}$ is semisimple, Whitehead's Lemma implies that ρ: $\mathfrak{b} \to \tilde{\mathfrak{b}}$ factors through σ, i.e., that we can find a Lie algebra homomorphism ϕ': $\mathfrak{b} \to \mathfrak{h}_0 \subseteq \mathfrak{h}$ such that $\sigma \circ \phi' = \rho$.

Because $\tau|\mathfrak{h}_0 = \Psi \circ \sigma$ and $\sigma \circ \phi' = \rho$, it follows, for all $X \in \mathfrak{b}$, that

$$[\mathrm{ad}_{\mathfrak{h}}(\phi'(X))]|\mathcal{V} = \tau(\phi'(X)) = \Psi(\sigma(\phi'(X))) = \Psi(\rho(X)) = \Psi(\tilde{X}).$$

Therefore, for all $X \in \mathfrak{b}$, for all $U \in V$, we have

$$[\phi'(X), \psi(U)] = [\mathrm{ad}_{\mathfrak{h}}(\phi'(X))][\psi(U)] = [\Psi(\tilde{X})][\psi(U)] = \psi(\tilde{X}U),$$

as desired. ∎

Note that the significant difference between (7.1) and (7.2) is that ϕ: $W \to \mathfrak{h}$ is simply a linear map, whereas ϕ': $\mathfrak{b} \to \mathfrak{h}$ is a Lie algebra homomorphism. The main content of (7.2) is that ρ: $\mathfrak{b} \to \mathfrak{gl}(V)$ is isomorphic to a subrepresentation of $\mathrm{ad}_{\mathfrak{h}} \circ \phi'$: $\mathfrak{b} \to \mathfrak{gl}(\mathfrak{h})$. Thus, (3) of Lemma 7.1 asserts that if ψ is injective, if $\tilde{\mathfrak{b}}$ is semisimple and if $\rho|W$ is isomorphic to a subrepresentation of $(\mathrm{ad}_{\mathfrak{h}} \circ \phi)|W$, then W can be replaced by the Lie algebra it generates, and ϕ can be replaced by a Lie algebra homomorphism. Even if $\tilde{\mathfrak{b}}$ is not semisimple, we still get some information from (2), namely that $\tilde{\mathfrak{b}}$ is a subquotient of $\mathfrak{h}$.

The essential content of the next lemma is that $\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ is not a Lie subalgebra of $\mathfrak{so}(1, d-1) \ltimes \mathbb{R}^d$.

LEMMA 7.2: *Let $d \geq 1$ be an integer. Let $\mathfrak{h} = \mathfrak{so}(1, d-1) \ltimes \mathbb{R}^d$. For all $X \in \mathfrak{sl}_2(\mathbb{R})$, let $\tilde{X}$: $\mathbb{R}^2 \to \mathbb{R}^2$ denote the linear map corresponding to X. Assume that ϕ: $\mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{h}$ is a Lie algebra homomorphism and that ψ: $\mathbb{R}^2 \to \mathfrak{h}$ is a nonzero linear map. Then there exist $X \in \mathfrak{sl}_2(\mathbb{R})$ and $U \in \mathbb{R}^2$ such that $[\phi(X), \psi(U)] \neq \psi(\tilde{X}U)$.*

Proof: Assume for a contradiction that, for all $X \in \mathfrak{sl}_2(\mathbb{R})$ and $U \in \mathbb{R}^2$, we have

$$[\phi(X), \psi(U)] = \psi(\tilde{X}U). \tag{7.5}$$

Since ψ is nonzero, $\mathfrak{sl}_2(\mathbb{R})$ is simple, and the standard representation of $\mathfrak{sl}_2(\mathbb{R})$ on $\mathbb{R}^2$ is irreducible, we conclude that both ψ and ϕ are injective. Thus the standard representation of $\mathfrak{sl}_2(\mathbb{R})$ on $\mathbb{R}^2$ is a subrepresentation of $\mathrm{ad}_{\mathfrak{h}} \circ \phi$: $\mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{gl}(\mathfrak{h})$. Since $\mathfrak{so}(1,0) = \{0\}$ and $\mathfrak{so}(1,1)$ are both Abelian, we conclude that $d \geq 3$.

Let $\mathfrak{b} := \psi(\mathfrak{sl}_2(\mathbb{R}))$. Using (7.5) and the irreducibility of the standard representation of $\mathfrak{sl}_2(\mathbb{R})$, we conclude that $\phi \neq 0$. Since $\mathfrak{sl}_2(\mathbb{R})$ is simple, ϕ: $\mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{b}$ is a Lie algebra isomorphism.

Define Q: $\mathbb{R}^d \to \mathbb{R}$ by $Q(x_1, \ldots, x_d) = 2x_1x_d + x_2^2 + \cdots + x_{d-1}^2$. We may assume that $\mathfrak{h} = \mathfrak{so}(Q) \ltimes \mathbb{R}^d$. Because $\mathfrak{b}$ is semisimple, there is an automorphism A: $\mathfrak{h} \to \mathfrak{h}$ such that $A(\mathfrak{b}) \subseteq \mathfrak{so}(Q)$. Replacing $\mathfrak{b}$ by $A(\mathfrak{b})$, ϕ by $A \circ \phi$ and ψ by $A \circ \psi$, we may assume that $\mathfrak{b} \subseteq \mathfrak{so}(Q)$.

Let $\mathfrak{a}$ be the maximal split torus in $\mathfrak{so}(Q)$ consisting of all diagonal matrices in $\mathfrak{so}(Q)$. Let $\mathfrak{a}_0$ be the maximal split torus in $\mathfrak{sl}_2(\mathbb{R})$ consisting of all diagonal matrices of trace zero. Then $\phi(\mathfrak{a}_0)$ is a maximal split torus in $\mathfrak{so}(Q)$, so there exists an automorphism A': $\mathfrak{h} \to \mathfrak{h}$ such that $A'(\phi(\mathfrak{a}_0)) = \mathfrak{a}$. Replacing $\mathfrak{b}$ by $A'(\mathfrak{b})$, ϕ by $A' \circ \phi$ and ψ by $A' \circ \psi$, we may assume that $\mathfrak{a} \subseteq \mathfrak{b}$ and that $\phi(\mathfrak{a}_0) = \mathfrak{a}$.

Let $J_0 \in \mathfrak{sl}_2(\mathbb{R})$ be the diagonal matrix with 1 in the $(1,1)$ entry and with -1 in the $(2,2)$ entry. Let $J \in \mathfrak{so}(Q)$ be the matrix with 1 in the $(1,1)$ entry, with -1 in the (d,d) entry and with 0s elsewhere.

Since $\phi(\mathfrak{a}_0) = \mathfrak{a} = \mathbb{R}J$, we conclude that $\phi(J_0) = \lambda J$ for some $\lambda \in \mathbb{R}$.

We calculate that $\mathrm{ad}_{\mathfrak{h}}(J)$: $\mathfrak{h} \to \mathfrak{h}$ has eigenvalues 1, 0 and -1 and that $\mathrm{ad}_{\mathfrak{sl}_2(\mathbb{R})}(J_0)$: $\mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})$ has eigenvalues 2, 0 and -2.

Since the representation $\mathrm{ad}_{\mathfrak{h}} \circ \phi$: $\mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{gl}(\mathfrak{h})$ contains both $\mathrm{ad}_{\mathfrak{sl}_2(\mathbb{R})}$ and the standard representation of $\mathfrak{sl}_2(\mathbb{R})$ on $\mathbb{R}^2$ as subresentations, we conclude that $\mathrm{ad}_{\mathfrak{h}}(\phi(J_0))$ has nontrivial eigenspaces with eigenvalues 2, 1, 0, -1 and 2. This contradicts the fact that $\mathrm{ad}_{\mathfrak{h}}(J)$ has eigenvalues 1, 0 and -1. ∎

8. Jets of vector fields of isometric actions

LEMMA 8.1: *Let g and h be quadratic differentials near the origin in a vector space, and let X and Y be vector fields near the origin. Assume that g and h agree to order one at zero and that X and Y agree to order one at zero. Then $L_X g - L_Y h$ vanishes at zero.*

Proof: Since g and h agree to order one at zero, $L_X(g-h)$ vanishes at zero, so it suffices to show that $L_X h = L_Y h$ at zero.

Fix vector fields Z and W near zero. We wish to show that

$$(L_X h)(Z,W) = (L_Y h)(Z,W)$$

at zero. Since X and Y agree to order one at zero, we get

$$X(h(Z,W)) = Y(h(Z,W)) \quad \text{and} \quad L_X Z = L_Y Z \quad \text{and} \quad L_X W = L_Y W$$

at zero. As $(L_X h)(Z,W) = X(h(Z,W)) - h(L_X Z, W) - h(Z, L_X W)$ and $(L_Y h)(Z,W) = Y(h(Z,W)) - h(L_Y Z, W) - h(Z, L_Y W)$, the result follows. ∎

Recall that (M,γ) is a pseduo-Riemannian manifold.

LEMMA 8.2: *Let $m_0 \in M$. Let N be a neighborhood of zero in $T_{m_0}M$ such that $\exp_{m_0}|N$ is a diffeomorphism onto a neighborhood M_0 of m_0. Let $\hat{\gamma} := \exp^*_{m_0}(\gamma)$. Let $\overline{\gamma}$ denote the flat pseudo-Riemannian metric on $T_{m_0}M$ corresponding to the inner product γ_{m_0} on $T_{m_0}M$. Then $\hat{\gamma}$ and $\overline{\gamma}$ agree to order one at zero.*

Proof: Fix $X, Y \in T_0(T_{m_0}M)$. Let $\check{X}, \check{Y} \in T_{m_0}M$ be the corresponding elements, under the identification of $T_0(T_{m_0}M)$ with $T_{m_0}M$. Then

$$\hat{\gamma}(X,Y) = \gamma(\check{X},\check{Y}) = \overline{\gamma}_{m_0}(X,Y).$$

Thus $\hat{\gamma} = \overline{\gamma}$ at zero.

Fix vector fields X, Y, Z in $T_{m_0}M$ defined near zero. It now suffices to show that $(L_Z\hat{\gamma})(X,Y) = (L_Z\overline{g})(X,Y)$ at zero.

Since $\hat{\gamma} = \overline{\gamma}$ at zero, we have $\hat{\gamma}(L_Z X, Y) = \overline{\gamma}(L_Z X, Y)$ at zero and $\hat{\gamma}(X, L_Z Y) = \overline{\gamma}(X, L_Z Y)$ at zero. Now

$$\begin{aligned} (L_Z\hat{\gamma})(X,Y) &= Z(\hat{\gamma}(X,Y)) - \hat{\gamma}(L_Z X, Y) - \hat{\gamma}(X, L_Z Y), \\ \text{and} \quad (L_Z\overline{\gamma})(X,Y) &= Z(\overline{\gamma}(X,Y)) - \overline{\gamma}(L_Z X, Y) - \overline{\gamma}(X, L_Z Y). \end{aligned}$$

It therefore suffices to show that $Z(\hat{\gamma}(X,Y)) = Z(\overline{\gamma}(X,Y))$ at zero.

Let $\hat{\nabla}$ be the Levi-Civita connection on $T_{m_0}M$ defined near zero corresponding to $\hat{\gamma}$ and let $\overline{\nabla}$ be the Levi-Civita connection on $T_{m_0}M$ corresponding to $\overline{\gamma}$. We now wish to show that

$$\hat{\gamma}(\hat{\nabla}_Z X, Y) + \hat{\gamma}(X, \hat{\nabla}_Z Y) = \overline{\gamma}(\overline{\nabla}_Z X, Y) + \overline{\gamma}(X, \overline{\nabla}_Z Y)$$

at zero.

Since $\hat\gamma = \overline\gamma$ at zero, we have $\hat\gamma(\hat\nabla_Z X, Y) = \overline\gamma(\hat\nabla_Z X, Y)$ at zero and $\hat\gamma(X, \hat\nabla_Z Y) = \overline\gamma(X, \hat\nabla_Z Y)$ at zero. It therefore suffices to show that $\hat\nabla_Z X = \overline\nabla_Z X$ at zero and $\hat\nabla_Z Y = \overline\nabla_Z Y$ at zero.

Let $n := \dim M$. Let $e_1, \dots, e_n$ be a basis for $T_{m_0}M$ and let $E_1, \dots, E_n$ be the corresponding constant vector fields on $T_{m_0}M$.

By a result that appears in a variety of references (e.g., [KN63, Proposition 8.4, p. 149, in Chapter III]), for all $i, j \in \{1, \dots, n\}$, we have that $\hat\nabla_{E_i} E_j$ vanishes at zero. Moreover, since $\overline\gamma$ is constant and hence flat, it follows, for all $i, j \in \{1, \dots, n\}$, that $\overline\nabla_{E_i} E_j = 0$.

Fix $i, j \in \{1, \dots, n\}$ and fix two functions f, g on $T_{m_0}M$. Then $\hat\nabla_{fE_i}(gE_j) = f(E_i g)E_j = \overline\nabla_{fE_i}(gE_j)$ at zero.

For any vector field R on $T_{m_0}M$, there exist functions $h_1, \dots, h_n$ on $T_{m_0}M$ such that $R = \sum h_i E_i$.

So, for any two vector fields P, Q on $T_{m_0}M$, we have $\hat\nabla_P Q = \overline\nabla_P Q$ at zero. In particular, $\hat\nabla_Z X = \overline\nabla_Z X$ at zero, and $\hat\nabla_Z Y = \overline\nabla_Z Y$ at zero, as desired. ∎

The next seven lemmas follow from straightforward computations and from Taylor's Theorem.

LEMMA 8.3: *Let X be a vector field defined near zero in a vector space. Let f be a function defined near zero vanishing to order k at zero. Then fX vanishes to order k at zero.*

LEMMA 8.4: *Let S and T be two vector fields defined near zero in a vector space. If S and T vanish at zero, then $[S, T]$ vanishes at zero.*

LEMMA 8.5: *Let S, T and U be vector fields defined near zero in a vector space. If S and T agree to order one at zero and U vanishes at zero, then $[S, U]$ and $[T, U]$ agree to order one at zero.*

LEMMA 8.6: *Let X be a vector field defined on a neighborhood of zero in a real vector space V. Then for any $k > 0$, there exist vector fields $X_0, \dots, X_k$ and R on V such that X_i is homogeneous of degree i, R vanishes to order k at zero, and $X = X_0 + \cdots + X_k + R$ on a neighborhood of zero.*

LEMMA 8.7: *Let C, C', L and L' be vector fields defined near zero in a vector space. Assume that C and C' are constant and that L and L' are linear. Then*

1. *$[L', C]$ is constant and $[L', L]$ is linear; and*
2. *if $C + L$ and $C' + L'$ agree to order one at zero, then $C = C'$ and $L = L'$.*

LEMMA 8.8: *Let L be a linear vector field in a vector space. Let B be a constant quadratic differential. Then $L_L B$ is constant.*

LEMMA 8.9: *Let V be a vector space with a quadratic form B. Let $\overline{B}$ denote the constant quadratic differential on V corresponding to B. Let C and L be vector fields, such that C is constant and L is linear. Assume that $L_{C+L}(\overline{B})$ vanishes at zero. Then $L \in \mathfrak{so}(B)$.*

Recall that G is a connected Lie group acting locally faithfully by isometries of a pseudo-Riemannian manifold (M, γ). Fix $m_0 \in M$. Let N be a starlike neighborhood of 0 in $T_{m_0}M$ such that $\exp_{m_0}$ is defined on N and such that $\exp_{m_0}$ is a diffeomorphism of N onto a neighborhood M_0 of m_0 in M.

Recall that $G_{m_0} := \mathrm{Stab}_G(m_0)$. Let $\hat{\gamma} := \exp^*_{m_0}(\gamma)$. Let $\overline{\gamma}$ denote the flat Lorentz metric on N coming from the Minkowski form γ_{m_0}.

For $X \in \mathfrak{g}$, recall that X_M is the vector field on M corresponding to X. For $X \in \mathfrak{g}$, let $\hat{X} := (\exp^{-1}_{m_0})_*(X_M)$. For $X \in \mathfrak{g}$, let $\hat{X}_C$, $\hat{X}_L$, $\hat{X}_Q$ and $\hat{X}_R$ be vector fields on $T_{m_0}M$ such that $\hat{X}_C$ is constant, $\hat{X}_L$ is linear, $\hat{X}_Q$ is quadratic, $\hat{X}_R$ vanishes to order two at zero and $\hat{X} = \hat{X}_C + \hat{X}_L + \hat{X}_Q + \hat{X}_R$.

LEMMA 8.10: *Assume that the action of G on M is by isometries.*

1. *For all $X \in \mathfrak{g}$, we have $\hat{X}_L \in \mathfrak{so}(\gamma_{m_0})$.*
2. *For all $U \in \mathfrak{g}$, if $\hat{U}_C = 0$, then $\hat{U} = \hat{U}_L$.*
3. *For all $X, U, T \in \mathfrak{g}$, if $\hat{U}_C = 0$ and if $T = [X, U]$, then*

$$\hat{T}_C = [\hat{X}_C, \hat{U}_L] \quad \text{and} \quad \hat{T}_L = [\hat{X}_L, \hat{U}_L].$$

Proof of Lemma 8.10: Proof of 1: By definition of $\hat{X}_C$ and $\hat{X}_L$, we know that $\hat{X}$ and $\hat{X}_C + \hat{X}_L$ agree to order one at zero. By Lemma 8.2, we see that $\hat{\gamma}$ and $\overline{\gamma}$ agree to order one at zero. By Lemma 8.1, since $L_{\hat{X}}\hat{\gamma}$ vanishes on N, we conclude that $L_{\hat{X}_C+\hat{X}_L}\overline{\gamma}$ vanishes at zero. Thus, by Lemma 8.9, $\hat{X}_L \in \mathfrak{so}(\gamma_{m_0})$.

Proof of 2: Since $\hat{U}_C = 0$, it follows that U_M vanishes at m_0. So, for all $t \in \mathbb{R}$, we have $(\exp tU)m_0 = m_0$, i.e., $\exp tU \in G_{m_0}$.

Naturality of the Riemannian exponential map implies that $\hat{U}$ is the restriction to N of the vector field of the flow

$$(t, v) \mapsto (\exp tU)_*(v) \colon \mathbb{R} \times T_{m_0}M \to T_{m_0}M.$$

Since this flow is a flow by linear transformations, $\hat{U}$ is linear, i.e., $\hat{U} = \hat{U}_L$.

Proof of 3: By definition of $\hat{X}_C$ and $\hat{X}_L$, we know that $\hat{X}$ and $\hat{X}_C + \hat{X}_L$ agree to order one at zero. Similarly, $\hat{T}$ and $\hat{T}_C + \hat{T}_L$ agree to order one at zero.

Since $\hat{U}_C = 0$, it follows that $\hat{U}$ vanishes at zero. By Lemma 8.5,

$$[\hat{X}_C + \hat{X}_L, \hat{U}] \quad \text{and} \quad [\hat{X}, \hat{U}]$$

agree to order one at zero. Because $T = [X, U]$, we have $\hat{T} = [\hat{X}, \hat{U}]$. Thus

$$[\hat{X}_C + \hat{X}_L, \hat{U}] \quad \text{and} \quad \hat{T}_C + \hat{T}_L$$

agree to order one at zero.

By (2) of Lemma 8.10, $\hat{U} = \hat{U}_L$. Thus

$$[\hat{X}_C + \hat{X}_L, U] = [\hat{X}_C, \hat{U}_L] + [\hat{X}_L, \hat{U}_L].$$

It follows that

$$[\hat{X}_C, \hat{U}_L] + [\hat{X}_L, \hat{U}_L] \quad \text{and} \quad \hat{T}_C + \hat{T}_L$$

agree to order one at zero.

By (1) of Lemma 8.7, $[\hat{X}_C, \hat{U}_L]$ is constant and $[\hat{X}_L, \hat{U}_L]$ is linear. We conclude from (2) of Lemma 8.7 that $\hat{T}_C = [\hat{X}_C, \hat{U}_L] = [X_C, U_L]$ and that $\hat{T}_L = [\hat{X}_L, \hat{U}_L] = [\hat{X}_L, \hat{U}_L]$. ∎

9. Restriction on stabilizers

Let $n \geq 2$ be an integer.

Assume that $G = \mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ and that (M, γ) is Lorentz. Recall that G acts locally faithfully by isometries of (M, γ). Let $m_0 \in M$. Recall that $G_{m_0} := \mathrm{Stab}_G(m_0)$. For all $X \in \mathfrak{sl}_n(\mathbb{R})$, let $\tilde{X}\colon \mathbb{R}^n \to \mathbb{R}^n$ denote the corresponding linear map.

Fix a starlike neighborhood N of $0 \in T_{m_0}M$ such that $\exp_{m_0}$ is defined on N and $\exp_{m_0}$ is a diffeomorphism of N onto a neighborhood M_0 of m_0 in M.

For all $X \in \mathfrak{g}$, let $\hat{X} := (\exp_{m_0}^{-1})_*(X_M)$.

Let $\hat{\gamma} := \exp_{m_0}^*(g)$. Let $\overline{\gamma}$ denote the flat Lorentz metric on N coming from the Minkowski form γ_{m_0}.

By Lemma 8.6, we can, for each $X \in \mathfrak{sl}_n(\mathbb{R}) \ltimes \mathbb{R}^n$, choose vector fields $\hat{X}_C$, $\hat{X}_L$ and $\hat{X}_R$ on $T_{m_0}M$ such that $\hat{X}_C$ is constant, $\hat{X}_L$ is linear, $\hat{X}_R$ vanishes to order one at zero, and $\hat{X} = \hat{X}_C + \hat{X}_L + \hat{X}_R$.

In Lemma 9.1 below, note that ϕ is only a linear map; there is no reason to believe that it is a Lie algebra homomorphism.

LEMMA 9.1: *Suppose that* $\mathbb{R}^n \subseteq G_{m_0}$. *Then there exist linear maps* $\phi\colon \mathfrak{sl}_n(\mathbb{R}) \to \mathfrak{so}(\gamma_{m_0})$ *and* $\psi\colon \mathbb{R}^n \to \mathfrak{so}(\gamma_{m_0})$ *such that* ψ *is injective and, for all* $X \in \mathfrak{sl}_n(\mathbb{R})$, *for all* $U \in \mathbb{R}^n$, *we have*

$$[\phi(X), \psi(U)] = \psi(\tilde{X}U).$$

Proof: Define $\phi\colon \mathfrak{sl}_n(\mathbb{R}) \to \mathfrak{gl}(T_{m_0}M)$ and $\psi\colon \mathbb{R}^n \to \mathfrak{gl}(T_{m_0}M)$ by

$$\phi(X) = \hat{X}_L \quad \text{and} \quad \psi(U) = \hat{U}_L.$$

By (1) of Lemma 8.10, $\phi(\mathfrak{sl}_n(\mathbb{R})) \subseteq \mathfrak{so}(\gamma_{m_0})$ and $\psi(\mathbb{R}^n) \subseteq \mathfrak{so}(\gamma_{m_0})$.

For all $U \in \mathbb{R}^n$, we have $U \in \mathfrak{g}_{m_0}$, so, by (1) of Lemma 8.10, $\hat{U}_C = 0$. By (2) of Lemma 8.10, we know that $\psi(U) = \hat{U}$.

If $\psi(U) = 0$, then $\hat{U} = 0$, so the Killing vector field U_M vanishes on M_0. If a Killing vector field on a connected pseudo-Riemannian manifold vanishes on a nonempty open set, then it vanishes everywhere. We conclude, for all $U \in \mathbb{R}^n$, that if $\psi(U) = 0$, then $U_M = 0$ which implies (by local faithfulness of the action) that $U = 0$. Therefore $\psi\colon \mathbb{R}^n \to \mathfrak{so}(\gamma_{m_0})$ is injective.

Fix $X \in \mathfrak{sl}_n(\mathbb{R})$ and $U \in \mathbb{R}^n$ and let $T := [X, U] = \tilde{X}U$. Then (3) of Lemma 8.10 implies that $[\hat{X}_L, \hat{U}_L] = \hat{T}_L$, which proves the Lemma. ∎

LEMMA 9.2: *Assume that* $n = 2$. *Then* $\mathbb{R}^2$ *cannot be contained* G_{m_0}.

Proof: Assume for a contradiction that $\mathbb{R}^2 \subseteq G_{m_0}$. Let ρ be the standard representation of $\mathfrak{sl}_2(\mathbb{R})$ on $V := \mathbb{R}^2$. Set $W := \mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{h} := \mathfrak{so}(\gamma_{m_0})$. Let ϕ and ψ be as in Lemma 9.1.

Let $d := \dim M$. Then $\mathfrak{h}$ is isomorphic to $\mathfrak{so}(1, d-1)$. Choose ϕ' as in (3) of Lemma 7.1. Replacing ϕ by ϕ' in the statement of Lemma 7.2, we arrive at a contradiction. ∎

10. Conclusion

THEOREM 10.1: *Let* $n \geq 3$ *be an integer. Let* $G = \mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ *act locally faithfully by isometries of a connected Lorentz manifold* M. *Then the action of* G *on* M *is proper.*

Proof: Suppose that the G-action on M is nonproper. We wish to obtain a contradiction.

By Lemma 6.1, there exists $m_0 \in M$ such that $\mathbb{R}^n$ is isotropic at m_0. By Lemma 4.1, we conclude that the stabilizer G_{m_0} in G of m_0 contains a codimension one subgroup of $\mathbb{R}^n$; as $n \geq 3$, we see that G_{m_0} contains a two-dimensional subspace S of $\mathbb{R}^n$.

Let S' be any vector space complement to S in $\mathbb{R}^n$. Then $SL(S)$ injects into $SL(S) \times SL(S')$, which, in turn, injects into $SL_n(\mathbb{R})$. Further, S is a subset of $\mathbb{R}^n$. These injections of $SL(S)$ into $SL_n(\mathbb{R})$ and of S into $\mathbb{R}^n$ combine to create an injection of $SL(S) \ltimes S$ into G.

Since S is two-dimensional, $SL(S) \ltimes S$ is isomorphic to $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$. We therefore obtain an injective homomorphism $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2 \to G$ such that $\mathbb{R}^2$ maps onto S. Then $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ acts on M and there exists $m_0 \in M$ such that $\mathbb{R}^2 \subseteq G_{m_0}$.

This contradicts Lemma 9.2. ∎

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